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ENGAGEMENT MODEL

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1.0 INTRODUCTION

Attempts to mathematically model engagements between Naval forces, infantry, tanks, airplanes and the like have to a great extent utilized the well-known Lanchester equations of combat. Though these equations are relatively easy to solve, and lend themselves well to analog methods of solution, they do have one major disadvantage in that they (the equations) are deterministic and the engagement stochastic. On the premise that a stochastic process is better represented by a stochastic model, this paper presents one formulation believed to be of general nature independent from the physical characteristics of and environment surrounding the participants of the engagement. The model is a function only of the combat potentials of the participants which may be abstracted, in a mathematical sense, from the scenario. The model is designed to function as the *core* of an engagement model independent from the surrounding *superstructure*. A naval combat model would differ from a land engagement model only in that portion which computes doctrine, maneuver, intelligence and other required *managerial* functions.

The stochastic process is assumed Discrete Markov in nature wherein each discrete "state" represents a certain condition of destroyed and existing participants. This assumption by no means excludes a time-varying process or heterogeneous force composition. The model is an extension of existing models which have been limited to *constant-coefficients* (time-invariance) and/or homogeneity of forces. The solution is not closed form in nature and, therefore, requires the use of a high speed digital computer for implementation.

The model is incapable of representing such parameters as *morale* degradation because of a violation of the Markov assumption. However, path dependent functions can be modeled as a *path-average* function of time if so desired.

The following sections describe the formulation of the model and solution algorithms.

2.0 MODEL FORMULATION

The engagement is modeled by a Discrete Non-Homogeneous Markov Stochastic process. The following discussion presents the methodology and algorithms.

The engagement consists of two opposing forces designated Red and Blue. Define N_R as the number of Red participants (elements) and N_B as the number of Blue elements. An "event" is defined as the destruction of one element. A "state" is defined as a particular set of destroyed and existing elements so that the total number of states (N_S) is

$$N_S = 2^{N_B + N_R} - 1$$

excluding the state where all elements are destroyed. A transition from one state to another is the occurrence of an event (the state where all are destroyed is automatically excluded by this definition).

The following matrix displays the states.

State	B_1	B_2	B_i	$B_{N,B}$	R_1	R_2	R_j	$R_{N.R}$
1	1	1	$\cdots$	1	1	1	$\cdots$	1
2	0	1	$\cdots$	1	1	1	$\cdots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
"	"	"	"	"	"	"	"	"
S	"	"	"	"	"	"	"	"
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
"	"	"	"	"	"	"	"	"
N_S	1	0	$\cdots$	0	0	0	$\cdots$	0

where: B_i is the i^{th} Blue element

R_j is the j^{th} Red element

and a "1" implies existing element

a "0" implies destroyed element

Figure A-1 - State Matrix.

The problem is to find the probabilities of occurrence of each state as a function of time.

$$P_i(t) \equiv P \text{ (occurence of state i at time t)}, \quad i = 1, 2, \cdots, N_S$$

The probabilities $P(t)$ are a subset of a more inclusive set of probabilities $P_{i,j}(t, t_0)$ where $P_{i,j}(t, t_0)$ is defined as the probability of occurrence of state i at time t given the occurrence of state j at time t_0.

The $P_{i,j}(t, t_0)$ satisfy the following:

Define

$\overline{X}(t)$ as a Markov Stochastic Process

where

$\overline{X}(t) = i, \quad i = 1, 2, \cdots, N_S$

Then $P_{ij}(t, t_0) = P\{\overline{X}(t) = i | \overline{X}(t_0) = j\}$ satisfies the Chapman-Kolmogoroff equation:

$$\frac{\partial P_{ij}(t,t_0)}{\partial t} = \sum_{m=1}^{N_S} \frac{\partial P_{im}(t,t_0)}{\partial t} P_{mj}(t,t_0) \qquad (1)$$

Define $$q_i(t) \equiv \left. \frac{\partial P_{ii}(t,t_0)}{\partial t} \right|_{t_0 = t}$$

and $$q_{ij}(t) \equiv \left. \frac{\partial P_{ij}(t,t_0)}{\partial t} \right|_{t_0 = t} \quad , \ i \neq j$$

Equation (1) then becomes

$$\frac{\partial P_{ij}(t,t_0)}{\partial t} = -q_i(t) P_{ij}(t,t_0) + \sum_{\substack{m=1 \\ m \neq i}}^{N_S} q_{im}(t) P_{mj}(t,t_0) \qquad (2)$$

Defining $t_0 = 0$ and $j = 1$ (engagement starts at time zero with all elements existing) allows the following simplification of Equation (2):

$$\frac{\partial P_i(t)}{\partial t} = -q_i(t) P_i(t) + \sum_{\substack{m=1 \\ m \neq i}}^{N_S} q_{im}(t) P_m(t) \qquad (3)$$

Equation (3) is the model. The $q_i(t)$ and $q_{im}(t)$ are functions of element potential and fire doctrine derived as follows:

$$q_i(t) \approx -\frac{P_{ii}(t+\Delta t,t) - P_{ii}(t,t)}{\Delta t}$$

$$q_{ij}(t) \approx -\frac{P_{ij}(t+\Delta t,t) - P_{ij}(t,t)}{\Delta t}$$

Obviously $\quad P_{ii}(t,t) = 1$ and $P_{ij}(t,t) = 0$

Therefore: $\quad P_{ii}(t+\Delta t,t) = 1 - q_i(t)\Delta t$

and $\quad P_{ij}(t+\Delta t,t) = q_{ij}(t)\Delta t$

The $q_i(t)$ and $q_{ij}(t)$ are average transition rates between states and are referred to as transition intensities. A transition intensity model is required to determine these parameters. Since the transition intensity model is a function of fire doctrine, many variations are possible. One possible formulation is presented as follows:

If one Red element is paired off with one Blue element, probabilities of kill (P_k) may be computed for each element. Assuming that each element fires randomly according to a Poisson distribution with average fire rate F, q_i takes the form

$$q_i = F P_k .$$

If each element has the option of firing at any one of a number of opposing elements, the following model is proposed:

To avoid confusion about subscripts, the subscript notation below is adopted.

i refers to the i^{th} Blue element

j refers to the j^{th} Red element

S and $S*$ refer to states which represent particular sets of $\left(B_1, B_2, \cdots, B_{N_B} \quad R_1, R_2, \cdots, R_{N_R}\right)$

then

$$B_i(S) = \begin{cases} 0 \\ 1 \end{cases} \quad \text{Existence parameters: function of the state } S$$

$$R_j(S) = \begin{cases} 0 \\ 1 \end{cases} \quad \text{Existence parameters; function of the state } S$$

$P_{K,B_{ij}}$ = Probability of kill of i^{th} Blue by the j^{th} Red

$P_{K,R_{ji}}$ = Probability of kill of j$^{\text{th}}$ Red by the i$^{\text{th}}$ Blue

$F_{B,i}$ = Fire rate of the i^{th} Blue

$F_{R,j}$ = Fire rate of the j^{th} Red

Let

$$\alpha_{ij} = P_{K,B_{ij}} \; F_{R,j}$$

$$\beta_{ji} = P_{K,R_{ji}} \; F_{B,i}$$

For a particular state S there are two conditions on S^*, namely:

1) S^* such that

$$B_i(S) = B_i(S^*), \quad i = 1,2,\cdots,i^*-1,i^*+1,\cdots,N_R$$

$$R_j(S) = R_j(S^*), \quad j = 1,2,\cdots,N_R$$

$$B_i{}^*(S) = 0$$

$$B_i{}^*(S^*) = 1$$

then $$q_{SS^*} = \sum_{j=1}^{N_R} R_j(S)\alpha_{i^*j}\left\{\frac{\beta_{ji^*}\alpha_{i^*j}}{\sum_{i=1}^{N_B} B_i(S)\beta_{ji}\alpha_{ij} + \beta_{ji^*}\alpha_{i^*j}}\right\} \tag{4}$$

2) S^* such that

$$B_i(S) = B_i(S^*), \quad i = 1,2,\cdots,N_B$$
$$R_j(S) = R_j(S^*) \quad j = 1,2,\cdots,j^*-1,j^*+1,\cdots,N_R$$
$$R_{j^*}(S) = 0$$
$$R_{j^*}(S^*) = 1$$

then $$q_{SS^*} = \sum_{i=1}^{N_B} B_i(S)\beta_{j^*i}\left\{\frac{\alpha_{ij^*}\beta_{j^*i}}{\sum_{j=1}^{N_R} R_j(S)\alpha_{ij}\beta_{ji} + \alpha_{ij^*}\beta_{j^*i}}\right\} \tag{5}$$

and

$$q_s = \sum_{i=1}^{N_B} B_i \sum_{j=1}^{N_R} R_j \; \alpha_{ij} \left\{ \frac{\beta_{ji}\,\alpha_{ij}}{\sum_{m=1}^{N_B} B_m \beta_{jm} \alpha_{mj}} \right\} + \sum_{j=1}^{N_R} R_j \sum_{i=1}^{N_B} B_i \; \beta_{ji} \left\{ \frac{\alpha_{ij}\,\beta_{ji}}{\sum_{m=1}^{N_R} R_m \alpha_{im} \beta_{mi}} \right\} \qquad (6)$$

The above formulation assumes a fire doctrine wherein element fire potential is distributed in such a fashion as eliminates the greatest immediate threat to the elements.

Any justifiable formulation for the transition intensities is acceptable so long as the total force fire potential is not exceeded.

3.0 METHOD OF SOLUTION

The solution is numerical, to be implemented on a high speed large capacity digital computer.

Inputs to the model are (1) P_K as a function of time and (2) fire rate F which also can vary with time for each element in each force.

The q_S and q_{SS^*} are functions of P_K and F which are inputs, as well as $R_j(S)$ and $B_i(S)$.

Storage of the entire transition intensity matrix (N_S x N_S) is impractical and quite inefficient (the matrix is almost entirely zero). An algorithm has been developed which requires the computation of only those non-zero elements in the matrix reducing the storage requirements by orders of magnitude with no increase in computer time.

Examination of the matrix in Figure A-1 shows that only transitions from state S^* to state S where $S^* \leq S$ are allowed. Given state S, the $R_j(S)$, $B_i(S)$ and all allowable values for S^* and the associated i^* (or j^*) are computed as follows:

If the binary representation of S is

$$S = \sum_{i=1}^{N_B} X_i \, 2^{i-1} + \sum_{j=N_{B+1}}^{N_B+N_R} Y_j \, 2^{j-1} \tag{7}$$

then $B_i(S) \equiv X_i \, , \; i = 1,2,\cdots,N_B$

and $R_j(S) \equiv Y_{j+N_B} \, , \; j = 1,2,\cdots,N_R$

where the X_i and Y_j are computed using the standard techniques

The number of states (N_T) from which a transition may take place into state S are computed as follows:

1) $\sum B_i \geq 1$ and $\sum R_j \geq 1$

$$N_T = N_B + N_R - \sum B_i - \sum R_j \tag{8}$$

2) $\sum B_i = 0 \, , \quad N_T = N_B$ (9)

3) $\sum R_j = 0 \, , \quad N_T = N_R$ (10)

If either condition 2) or 3) is satisfied, state S is known as a terminal state in that no further transitions out of S may take place.

The values for $S*$ are computed as follows:

$$S* = S - 2^{i*-1}$$

where $B_{i*}(S) = 0$ and $\sum \mathrm{R_j} \geq 1$

Or

$$S* = S - 2^{N_B + j* - 1}$$

where $R_{j*}(S) = 0$ and $\sum B_i \geq 1$

The values for q_{SS*} and q_s are computed from Equations (4) , (5) and (6) in Sec. 2.0. Equation (3) Sec. 2.0 is solved numerically using standard techniques. For each time increment Δt, the number of iterations (N_c) is

$$N_c = \sum_{S=1}^{N_S} N_T(S) \ , \qquad \text{see Equations (2),(3) and (4)}$$

Storage must be provided for the N_S values of $P_K(t)$. No other major storage blocks are required.

4.0 TYPICAL APPLICATIONS

The described methodology has been applied in part to a Naval Surface Combat study and analysis of direct fire engagements (1 vs. 1) and (2 vs. 1), TACWAR #71 & #94 respectively. The methods provided the means to quantify such *unquantifiables* as the value of time spent on the objective for land combat engagements, optimum fire rates considering the attendant logistical burden versus expected combat potential, and the worth of speed in Naval engagements.

Integration of the methodology with the processes of tactical command provides a tool with which to calculate the rate of expenditure (τ) of combat potential.

Other applications in Chapter 8.7, an examination of ASW, and research into the Direct and Support Fire land combat missions.